

ON THE LENGTH OF A PARTIAL INDEPENDENT TRANSVERSAL IN A MATROIDAL LATIN SQUARE

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ABSTRACT. We suggest and explore a matroidal version of the Brualdi - Ryser conjecture about Latin squares. We prove that any $n \times n$ matrix, whose rows and Columns are bases of a matroid, has an independent partial transversal of length $\lceil 2n/3 \rceil$. We show that for any n , there exists such a matrix with a maximal independent partial transversal of length at most $n - 1$.

1. INTRODUCTION

A Latin Square of order n is an $n \times n$ array L with entries taken from the set $\{1, \dots, n\}$, where each entry appears exactly once in each row or column of L . A *partial transversal* of size k of a Latin square L is a subset of k different entries of L , where no two of them lie in the same row or column.

The maximal size of a partial transversal in L will be denoted here by $t(L)$ and the minimal size of $t(L)$, over all Latin squares L of order n , will be denoted by $T(n)$. It has been conjectured by Ryser [10] that $T(n) = n$ for every odd n and by Brualdi [4] (see also [2] p. 255) that $T(n) = n - 1$ for every even n . Although these conjectures are still unsettled, a consistent progress has been made towards its resolution: Koksma [8] proved that for $n \geq 3$, $T(n) \geq \lceil (2n+1)/3 \rceil$. This bound was improved by Drake [5] to $T(n) \geq \lceil 3n/4 \rceil$ for $n > 7$, and again by de Veris and Wieringa [3] who obtained a lower bound of $\lceil (4n-3)/5 \rceil$ for $n \geq 12$. Woollbright [14] showed that $T(n) \geq \lceil n - \sqrt{n} \rceil$. A similar result was obtained independently by Brouwer, de Vries and Wieringa [1]. Recently, Hatami and Shor [6] proved that $T(n) \geq n - O(\log^2 n)$. See also a recent comprehensive survey by Wanless [11].

The aim of this note is to suggest and explore a matroidal version of the Brualdi-Ryser conjectures. For basic texts on matroids the reader is referred to Welsh [12], Oxley [9] and White [13].

Definition 1.1. Let (M, S) be a matroid M on a ground set S . A *matroidal Latin square* (abbreviated *MLS*) of degree n over (M, S) is an $n \times n$ matrix A whose entries are elements of S , where each row or column of A is a base of M .

Notice that a matroidal Latin square reduces to a Latin square if M is a partition matroid. We mention that according to a well-known conjecture of Rota [7] every

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set of n bases of a matroid of rank n can be arranged to form an MLS of degree n so that its rows consist of the original bases.

Definition 1.2. *An independent partial transversal of an MLS A is an independent subset of entries of A where no two of them lie in the same row or column of A .*

We propose the following analogue of Brualdi's conjecture:

Conjecture 1.3. *Every MLS of degree n has an independent partial transversal of size $n - 1$.*

In view of Ryser's conjecture, it is natural to ask whether in Conjecture 1.3 an independent transversal of size n exists whenever n is odd. Theorem 3.1 asserts that this is not the case.

2. A LOWER BOUND FOR A MAXIMAL INDEPENDENT PARTIAL TRANSVERSAL

Let $A = (a_{ij})_{i,j=1}^n$ be an MLS of degree n over a matroid M . Let T be an independent partial transversal of size t . Without loss of generality we may assume that the elements of T are the first t elements of the main diagonal of A . That is

$$(2.1) \quad A = \left(\begin{array}{c|c} B & C \\ \hline D & E \end{array} \right)$$

where B , C , D and E are sub-matrices of A of dimensions $t \times t$, $t \times (n-t)$, $(n-t) \times t$ and $(n-t) \times (n-t)$ respectively, and T constitutes the main diagonal of B . If T is of maximal length, then $t \geq \lceil n/2 \rceil$. Otherwise $\dim(E) \geq n-t > t = \dim(T)$ and thus E would contain an element that is not spanned by T and hence can be added to T , contradicting the maximality of T . In order to show that $t \geq \lceil 2n/3 \rceil$ we shall need the following lemma:

Lemma 2.1. *Let X be a finite set and let $s > |X|/2$. Let X_1, \dots, X_s be a family of subsets of X , each of size at least s . Then there exists some X_i , all of whose elements appear in other subsets in the family.*

Proof. Let Y_1 be the set of elements in X that appear in exactly one of the subsets X_1, \dots, X_s and let Y_2 be the set of elements in X that appear in at least two of the subsets X_1, \dots, X_s . Let $k_1 = |Y_1|$ and $k_2 = |Y_2|$. Assume, by contradiction, that each X_i contains at least one element of Y_1 . Then $k_1 \geq s$ and thus

$$(2.2) \quad k_2 \leq |X| - k_1 \leq |X| - s < |X|/2$$

(since $s > |X|/2$). If, for some i , $|X_i \cap Y_1| = 1$ then $|X_i \cap Y_2| \geq s - 1$ and thus $k_2 \geq s - 1 > |X|/2 - 1$. It follows that $k_2 \geq |X|/2$, contradicting (2.2). It follows that for all i , $|X_i \cap Y_1| \geq 2$. Then $k_1 \geq 2s$ and thus $k_2 \leq |X| - k_1 \leq |X| - 2s < |X| - |X| = 0$, which is absurd. This proves the lemma. \square

Theorem 2.2. *Let A be an MLS of degree n over a matroid M . Then A contains an independent partial transversal of size $\lceil 2n/3 \rceil$.*

Proof. We use the notations from the beginning of Section 2. Since T is maximal, all the elements in the sub-matrix E are spanned by T . Let T_E be the minimal subset of T that spans E (this set is unique since T is independent.) Since $\dim(E) \geq n-t$ then $|T_E| \geq n-t$ and thus $|T \setminus T_E| \leq t - (n-t) = 2t - n$. Since each row of A is a base and all the elements of E are spanned by T , each row of the sub-matrix D contains a subset of size $n-t$ that complement T to a base. In particular,

each row of D contains at least $n - t$ elements that are not spanned by T . Let $X = \{1, \dots, t\}$ be the set of indices of the columns of D . For each of the $n - t$ rows in D we define a subset $X_i \subseteq X$, $i = t + 1, \dots, n$, in the following way: $j \in X_i$ if and only if the j th element of the i th row of A is not spanned by T . It follows that $|X_i| \geq n - t$ for all $i = t + 1, \dots, n$. Now assume, by contradiction, that $t < 2n/3$. Then $n - t > n/3 > t/2$. So we have a set X of size t and $n - t$ subsets X_{t+1}, \dots, X_n , each of size at least $n - t$, such that $n - t > t/2$. Let $s = n - t$. By Lemma 2.1 we conclude that there exists a subset X_i all of whose elements are contained in other subsets in the family X_{t+1}, \dots, X_n . This means that there is a row in D containing at least $n - t$ elements that are not spanned by T and for each such element there exists another element in the same column in D that is not spanned by T . It follows that D contains at least $n - t$ columns, each containing at least two elements that are not spanned by T . Since $t < 2n/3$ we have that $|T \setminus T_E| \leq 2t - n < n/3 < n - t$. So there exists $j \leq t$ such that (1) $a_{jj} \in T_E$ and (2) the j th column of D contains at least two elements that are not spanned in T . Let $x \in E$ be such that its support (i.e., its minimal spanning set) in T contains a_{jj} and let y and z be two elements in the j th column of D that are not spanned by T . We may assume that x and y are not in the same row (otherwise we take z instead of y). Since $T \cup \{y\}$ is independent, and the support of x in T contains a_{jj} , it follows that $T \setminus \{a_{jj}\} \cup \{y\}$ does not span x and thus $S \setminus \{a_{jj}\} \cup \{x, y\}$ is an independent partial transversal in A of length $t + 1$, contrary to the maximality of T . Thus t must be at least $\lceil 2n/3 \rceil$. \square

3. AN UPPER BOUND OF SIZE $n - 1$ FOR AN MLS OF DEGREE n

It is well known that for any even n there exist Latin squares of order n with no transversal of size n . The following theorem shows that for any n there exists an MLS of degree n with no independent transversal of size n .

Theorem 3.1. *Let v_1, v_2, \dots, v_n be a basis of a vectorial matroid of rank n . Then the matrix $A = (a_{ij})_{i,j=1}^n$, whose elements are $a_{ii} = v_i$, for $i = 1, \dots, n$, and $a_{ij} = v_i - v_j$, for $1 \leq i \neq j \leq n$, is an MLS of order n with no independent transversal of size n .*

Proof. We leave it to the reader to check that the rows and columns of A are independent. Let T be a transversal of size n in A . We show that T is not independent. If T does not contain elements of the main diagonal of A , then, since each row and column is represented exactly once among the elements of T , the sum of the elements of T is 0, and T is not independent. Thus we may assume that T meets the main diagonal exactly once. Let $a_{ii} = v_i \in T$. If $i = 1$ then the sum of the elements of $T - a_{11}$ is 0. If $i > 1$, then v_i is not spanned by T , so T is not a basis, and thus, is not independent. \square

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